

Research Article

An Extended Generalized q -Extensions for the Apostol Type Polynomials

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Through a modification on the parameters associated with generating function of the q -extensions for the Apostol type polynomials of order α and level m , we obtain some new results related to a unified presentation of the q -analog of the generalized Apostol type polynomials of order α and level m . In addition, we introduce some algebraic and differential properties for the q -analog of the generalized Apostol type polynomials of order α and level m and the relation of these with the q -Stirling numbers of the second kind, the generalized q -Bernoulli polynomials of level m , the generalized q -Apostol type Bernoulli polynomials, the generalized q -Apostol type Euler polynomials, the generalized q -Apostol type Genocchi polynomials of order α and level m , and the q -Bernstein polynomials.

1. Introduction

With the development of q -calculus in the mid-19th century, many authors made generalizations to special functions and polynomial families based on the q -analogs (cf. [1–7]). During this process, properties and relations have been demonstrated and contributed to solving different kinds of problems in other subjects (see, [8–10]).

In 2003, Natalini P. and Bernardini A. [11] introduced a class the polynomials $B_n^{[m-1]}(x)$, considering a class of Appell polynomials defined by using a generating function linked to the Mittag-Leffler function (see, [12, p. 204, Eq. (2)])

$$E_{1,m+1}(z) := \frac{z^m}{e^z - \sum_{h=0}^{m-1} (z^h/h!)}, \quad m \in \mathbb{N}. \quad (1)$$

Kurt B. [13] did the generalization of the Bernoulli $B_n^{[m-1,\alpha]}(x)$, Euler $E_n^{[m-1,\alpha]}(x)$, and Genocchi $G_n^{[m-1,\alpha]}(x)$ polynomials of order α and level m . Tremblay R. et al. [14, 15] defined the generalized Apostol-Bernoulli polynomials $B_n^{[m-1,\alpha]}(x; \lambda)$ and their properties. In [12] which has studied the unification of Bernoulli, Euler, and Genocchi polynomials they considered

the following Mittag-Leffler type function (see, [12, p. 209, Eq. (12)]):

$$E_{1,m+1}^{(c,a;\lambda;\mu;\nu)}(z) := \frac{(2^u z^\nu)^m}{\lambda c^z + \sum_{l=0}^{m-1} ((z \ln a)^l / l!)}, \quad (2)$$

$$m \in \mathbb{N}, \quad a, c \in \mathbb{R}^+, \quad \lambda, \mu, \nu \in \mathbb{C},$$

to define an extension of the generalized Apostol type polynomials in x , parameters c, a, μ, ν , order α , and level m through the following generating function:

$$(E_{1,m+1}^{(c,a;\lambda;\mu;\nu)}(z))^\alpha c^{xz} := \sum_{n=0}^{\infty} Q_n^{[m-1,\alpha]}(x; c, a; \lambda, \mu; \nu) \frac{z^n}{n!}, \quad (3)$$

where $m \in \mathbb{N}$, $a, c \in \mathbb{R}^+$, and $\alpha, \lambda, \mu, \nu \in \mathbb{C}$. The numbers were given by

$$Q_n^{[m-1,\alpha]}(c, a; \lambda, \mu; \nu) := Q_n^{[m-1,\alpha]}(0; c, a; \lambda, \mu; \nu). \quad (4)$$

Recently a new class of polynomials has been introduced in [16] and it provides a unification of three families of

polynomials through the following generating function (see, [16, p. 923, Eq. (3)]):

$$\left(\frac{2^{1-k} z^k}{\beta^b e_q^z - a^b} \right)^\alpha e_q^{xz} E_q^{yz} = \sum_{n=0}^{\infty} \mathcal{P}_{n,\beta,q}^{(\alpha)}(x, y; k, a, b) \frac{z^n}{[n]_q!}, \quad (5)$$

where $k \in \mathbb{N}_0$, $a, b \in \mathbb{R} \setminus \{0\}$, and $\alpha, \beta \in \mathbb{C}$. The author named it the unified q -Apostol-Bernoulli, Euler, and Genocchi polynomials of order α and proved some properties for these unification.

On the other hand, in the definition of the q -Mittag-Leffler function when $\alpha = 1$, $\beta = m + 1$, and $\gamma = 1$ lead us to (see, [17, p. 614, Eq. (1.5)])

$$E_{1,m+1}(z; q) := \frac{z^m}{e_q^z - \sum_{h=0}^{m-1} (z^h / [h]_q!)}, \quad m \in \mathbb{N}, \quad (6)$$

which corresponds to the q -analog of the generating function defined in (1) and with this, new research emerged about other polynomial families based on the q -analogs.

In [18], the authors introduced the generalized q -Apostol-Bernoulli polynomials, the generalized q -Apostol Euler polynomials, and generalized q -Apostol Genocchi polynomials in variable x, y , order α , and level m through the following generating functions, defined in a suitable neighborhood of $z = 0$ (see, [18, p. 2 Eq. (8), (9), (10)])

$$\left(\frac{z^m}{\lambda e_q^z - T_{m-1,q}(z)} \right)^\alpha e_q^{xz} E_q^{yz} \quad (7)$$

$$= \sum_{n=0}^{\infty} B_{n,q}^{[m-1,\alpha]}(x, y; \lambda) \frac{z^n}{[n]_q!},$$

$$\left(\frac{2^m}{\lambda e_q^z + T_{m-1,q}(z)} \right)^\alpha e_q^{xz} E_q^{yz} \quad (8)$$

$$= \sum_{n=0}^{\infty} E_{n,q}^{[m-1,\alpha]}(x, y; \lambda) \frac{z^n}{[n]_q!},$$

$$\left(\frac{2^m z^m}{\lambda e_q^z + T_{m-1,q}(z)} \right)^\alpha e_q^{xz} E_q^{yz} \quad (9)$$

$$= \sum_{n=0}^{\infty} G_{n,q}^{[m-1,\alpha]}(x, y; \lambda) \frac{z^n}{[n]_q!},$$

where $q, \alpha \in \mathbb{C}$, $m \in \mathbb{N}$, $0 < |q| < 1$, and $T_{m-1,q}(z) = e_q^z - \sum_{h=0}^{m-1} (z^h / [h]_q!)$.

Based on the previous result, we focus our attention on a new unification of the q -analog of the generalized Apostol type polynomials of order α and level m , defined in [18], by doing some modifications to the generating function linked to the q -Mittag-Leffler function (6) through new parameters following the same scheme or procedure applied by [12].

The paper is organized as follows. Section 2 contains some notations, definitions, and properties of the q -analogs and some results about q -analogs of the Apostol type polynomials. In Section 3, we introduce the unification q -analog of

the generalized Apostol type polynomials in x, y , parameters $\lambda, \mu, \nu \in \mathbb{C}$, order $\alpha \in \mathbb{C}$, and level $m \in \mathbb{N}$ and their algebraic and differential properties. Finally in the Section 4, we show relations between the q -analog of the generalized Apostol type polynomials and the q -Stirling numbers of the second kind, the generalized q -Bernoulli polynomials of level m , the generalized q -Apostol type Bernoulli polynomials, the generalized q -Apostol type Euler polynomials, the generalized q -Apostol type Genocchi polynomials of order α and level m , and the q -Bernstein polynomials.

2. Background and Previous Results

Throughout this paper, we use the following standard notions: $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, \mathbb{Z} denotes the set of integers, \mathbb{R} denotes the set of real numbers, and \mathbb{C} denotes the set of complex numbers. The q -numbers and q -factorial numbers are defined, respectively, by

$$[z]_q = \frac{1 - q^z}{1 - q} = \frac{q^z - 1}{q - 1}, \quad z \in \mathbb{C}, \quad q \in \mathbb{C} \setminus \{1\}, \quad q^z \neq 1, \quad (10)$$

$$[n]_q! = \prod_{k=1}^n [k]_q = [1]_q [2]_q [3]_q \dots [n]_q,$$

$$[0]_q! = 1, \quad n \in \mathbb{N}.$$

The q -shifted factorial is defined as

$$(a; q)_n = \prod_{j=0}^{n-1} (1 - q^j a), \quad n \in \mathbb{N}. \quad (11)$$

The q -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!}, \quad (n, k \in \mathbb{N}_0; 0 \leq k \leq n). \quad (12)$$

For more information about the q -standard definitions and properties see [8, 9, 19].

Furthermore, the q -binomial coefficient satisfies the following identity (see [10, p. 483, Eq. (41)]):

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q = \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n-j \\ k-j \end{bmatrix}_q \quad 0 \leq j \leq k. \quad (13)$$

The q -analog of the function $(x + y)^n$ is defined by

$$(x + y)_q^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(1/2)k(k-1)} x^{n-k} y^k, \quad n \in \mathbb{N}_0, \quad (14)$$

$$(1 - a)_q^n = (a; q)_n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(1/2)k(k-1)} (-1)^k a^k$$

$$= \prod_{j=0}^{n-1} (1 - q^j a).$$

The q -derivative of a function $f(z)$ is defined by

$$D_q f(z) = \frac{d_q f(z)}{d_q z} = \frac{f(qz) - f(z)}{(q-1)z}, \quad (15)$$

$$0 < |q| < 1, \quad 0 \neq z \in \mathbb{C}.$$

The q -analog of the exponential function is defined in two ways

$$e_q^z = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1 - (1-q)q^k z)}, \quad (16)$$

$$0 < |q| < 1, \quad z \in \mathbb{C}, \quad |z| < \frac{1}{|1-q|},$$

$$E_q^z = \sum_{n=0}^{\infty} \frac{q^{(1/2)n(n-1)} z^n}{[n]_q!} = \prod_{n=0}^{\infty} (1 + (1-q)q^n z),$$

$$0 < |q| < 1, \quad z \in \mathbb{C};$$

we can see that

$$e_q^z \cdot E_q^{-z} = 1, \quad (17)$$

$$e_q^{z+w} = e_q^z \cdot E_q^w. \quad (18)$$

Therefore,

$$D_q e_q^z = e_q^z, \quad (19)$$

$$D_q E_q^z = E_q^{qz}.$$

For any $t > 0$ (see, e.g., [19, p. 76, Eq (21.6)])

$$\Gamma_q(t) = \int_0^{\infty} x^{t-1} E_q^{-qx} d_q x \quad (20)$$

is called the q -gamma function.

The Jackson's q -gamma function is defined by (see [10, p. 490, Eq. (2)])

$$\Gamma_q(x) = \frac{(q; q)_{\infty}}{(q^x; q)_{\infty}} (1-q)^{1-x}, \quad 0 < |q| < 1. \quad (21)$$

In (21) we have

$$\Gamma_q(n+1) = \frac{(q; q)_{\infty}}{(q^{n+1}; q)_{\infty}} (1-q)^{-n} = (q; q)_n (1-q)^{-n} \quad (22)$$

$$= [n]_q!, \quad n \in \mathbb{N}.$$

For $\alpha, \beta, \gamma \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\beta) > 0$, and $\operatorname{Re}(\gamma) > 0$ y $|q| < 1$ the function $E_{\alpha, \beta}^{\gamma}(z; q)$ is defined as (see, e.g., [17, p. 614, Eq. (1.5)])

$$E_{\alpha, \beta}^{\gamma}(z; q) = \sum_{n=0}^{\infty} \frac{(q^{\gamma}; q)_n}{(q; q)_n} \frac{z^n}{\Gamma_q(\alpha n + \beta)}. \quad (23)$$

Notice that, when $\gamma = 1$ the previous equation is expressed as

$$E_{\alpha, \beta}(z; q) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(\alpha n + \beta)}. \quad (24)$$

Setting $\alpha = 1$ y $\beta = m + 1$, we can deduce

$$\sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(n + m + 1)} = \sum_{n=0}^{\infty} \frac{z^n}{[n + m]_q!} = \frac{1}{z^m} \sum_{h=m}^{\infty} \frac{z^h}{[h]_q!}$$

$$= \frac{(e_q^z - \sum_{h=0}^{m-1} (z^h / [h]_q!))}{z^m}.$$

The q -Stirling numbers of the second kind $S(n, k)_q$ are defined through the following expansion (see, e.g., [20, p. 173, Equ (5.18)]):

$$x^n = \sum_{k=0}^n S(n, k)_q (x)_{k, q}, \quad (26)$$

where

$$(x)_{k, q} = \prod_{n=0}^{k-1} (x - [n]_q). \quad (27)$$

Let $C[0, 1]$ denote the set of continuous function on $[0, 1]$. For any $f \in C[0, 1]$, the q - $\mathbb{B}_n(f; x)$ is called q -Bernstein operator of order n for f and is defined as (see, e.g., [21, p. 2, Eq (1.1)])

$$\mathbb{B}_n(f; x) = \sum_{r=0}^n f_r \begin{bmatrix} n \\ r \end{bmatrix}_q x^r \prod_{s=0}^{n-r-1} (1 - q^s x)$$

$$= \sum_{r=0}^n f_r B_r^n(x), \quad (28)$$

where $f_r = f([r]_q / [n]_q)$. For $n, r \in \mathbb{N}_0$, the q -Bernstein polynomials of degree n or q -Bernstein basis are defined by

$$B_r^n(x) = \begin{bmatrix} n \\ r \end{bmatrix}_q x^r \prod_{s=0}^{n-r-1} (1 - q^s x). \quad (29)$$

We know that $\sum_{k=0}^{n-j} B_k^{n-j}(x) = 1$, then

$$x^j = \sum_{k=0}^{n-j} \begin{bmatrix} n-j \\ k \end{bmatrix}_q x^{j+k} \prod_{t=0}^{n-j-k-1} (1 - q^t x), \quad (30)$$

and using the identity (13), we have (see [21, p. 6, Eq (2.3)])

$$x^j = \sum_{k=j}^n \begin{bmatrix} k \\ j \end{bmatrix}_q B_k^n(x). \quad (31)$$

Mahmudov N.I. [22] made a relation between the q -Bernstein basis with the q -Stirling numbers of the second kind and the q -Bernoulli polynomials of order α , $\alpha = k$ as follows:

$$B_k^n(x; q) = x^k \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q S(j, k; q) \mathfrak{B}_{n-j}^{(k)}(1, -x; q). \quad (32)$$

Proposition 1. For a fixed $m \in \mathbb{N}$, $n, k \in \mathbb{N}_0$, and $0 < |q| < 1$, let $\{B_n^{[m-1, \alpha]}(x, y; q)\}_{n=0}^\infty$ be the sequence of generalized q -Bernoulli polynomials in x, y of level m . Then the following identities are satisfied:

(1) [23, Lemma 10, Eq. (1)]

$$x^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[k]_q!}{[k+m]_q!} B_{n-k}^{[m-1, 1]}(x, 0; q). \quad (33)$$

(2) [23, Lemma 10, Eq. (2)]

$$y^n = \frac{1}{q^{(1/2)n(n-1)}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[k]_q!}{[k+m]_q!} B_{n-k}^{[m-1, 1]}(0, y; q). \quad (34)$$

Now, setting $k = n - k$, we have

$$y^n = \frac{[n]_q!}{q^{(1/2)n(n-1)} [n+m]_q!} \cdot \sum_{k=0}^n \begin{bmatrix} n+m \\ k \end{bmatrix}_q B_k^{[m-1, 1]}(0, y; q). \quad (35)$$

(3)

$$x^n = \sum_{k=0}^n \frac{[n]_q! B_k^{[m-1, 1]}(x, 0; q)}{[k]_q! \Gamma_q(n-k+m+1)}. \quad (36)$$

Proof (see (36)). Setting $\alpha = \lambda = 1$, $y = 0$ in (7) and using (25), we have

$$\begin{aligned} \frac{e_q^{xz}}{\sum_{n=0}^\infty (z^n / \Gamma_q(n+m+1))} &= \sum_{n=0}^\infty B_n^{[m-1, 1]}(x, 0; q) \frac{z^n}{[n]_q!} \\ e_q^{xz} &= \sum_{n=0}^\infty B_n^{[m-1, 1]}(x, 0; q) \frac{z^n}{[n]_q!} \sum_{n=0}^\infty \frac{z^n}{\Gamma_q(n+m+1)} \\ &= \sum_{n=0}^\infty \sum_{k=0}^n \frac{[n]_q! B_k^{[m-1, 1]}(x, 0; q)}{[k]_q! \Gamma_q(n-k+m+1)} \cdot \frac{z^n}{[n]_q!} \\ \sum_{n=0}^\infty \frac{x^n z^n}{[n]_q!} &= \sum_{n=0}^\infty \sum_{k=0}^n \frac{[n]_q! B_k^{[m-1, 1]}(x, 0; q)}{[k]_q! \Gamma_q(n-k+m+1)} \cdot \frac{z^n}{[n]_q!}. \end{aligned} \quad (37)$$

Comparing the coefficients of $z^n / [n]_q!$ we obtain

$$x^n = \sum_{k=0}^n \frac{[n]_q! B_k^{[m-1, 1]}(x, 0; q)}{[k]_q! \Gamma_q(n-k+m+1)}. \quad (38)$$

□

Notice that in (36), as $\Gamma_q(n-k+m+1) = [n-k+m]_q!$ we can get (33).

Based on the results of (2), (6), and (7)–(9) and following the methodology given in [12], we consider the following q -Mittag-Leffler type function:

$$E_{1, m+1}^{(\lambda; \mu; \nu)}(z; q) := \frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{h=0}^{m-1} (z^h / [h]_q!)}, \quad (39)$$

where $m \in \mathbb{N}$, $\lambda, \mu, \nu \in \mathbb{C}$.

3. The Polynomials $\mathcal{T}_n^{[m-1, \alpha]}(x, y; \lambda; \mu; \nu; q)$ and Their Properties

Definition 2. Let $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\alpha, \lambda, \mu, \nu, q \in \mathbb{C}$, and $0 < |q| < 1$; the q -analog of the generalized Apostol type polynomials in x, y , with parameters λ, μ, ν , order α , and level m is defined by means of the following generating function, in a suitable neighborhood of $z = 0$,

$$\begin{aligned} &\left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^\alpha e_q^{xz} E_q^{yz} \\ &= \sum_{n=0}^\infty \mathcal{T}_n^{[m-1, \alpha]}(x, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!}, \end{aligned} \quad (40)$$

where $|z| < |\log_q(-\lambda)|$ when $\lambda \in \mathbb{C} \setminus \{-1, 1\}$ and $1^\alpha := 1$. The numbers are given by

$$\begin{aligned} &\left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^\alpha \\ &= \sum_{n=0}^\infty \mathcal{T}_n^{[m-1, \alpha]}(\lambda; \mu; \nu; q) \frac{z^n}{[n]_q!}. \end{aligned} \quad (41)$$

Notice that

$$\mathcal{T}_n^{[m-1, \alpha]}(0, 0; \lambda; \mu; \nu; q) = \mathcal{T}_n^{[m-1, \alpha]}(\lambda; \mu; \nu; q). \quad (42)$$

Therefore,

$$\begin{aligned} \mathcal{T}_n^{[m-1, \alpha]}(x, 0; \lambda; \mu; \nu; q) &= \mathcal{T}_n^{[m-1, \alpha]}(x, \lambda; \mu; \nu; q), \\ \mathcal{T}_n^{[m-1, \alpha]}(0, y; \lambda; \mu; \nu; q) &= \mathcal{T}_n^{[m-1, \alpha]}(y, \lambda; \mu; \nu; q). \end{aligned} \quad (43)$$

We will use this notation $\mathcal{T}_n^{[m-1, \alpha]}(\lambda; \mu; \nu; q)$ instead of $\mathcal{T}_{n,q}^{[m-1, \alpha]}(\lambda; \mu; \nu)$ through the article. By comparing Definition 2 with (7)–(9), we have

$$\begin{aligned} (-1)^\alpha \mathcal{T}_n^{[m-1, \alpha]}(x, y; -\lambda; 0; 1; q) &= B_{n,q}^{[m-1, \alpha]}(x, y; \lambda), \\ \mathcal{T}_n^{[m-1, \alpha]}(x, y; \lambda; 1; y; q) &= E_{n,q}^{[m-1, \alpha]}(x, y; \lambda), \\ \mathcal{T}_n^{[m-1, \alpha]}(x, y; \lambda; 1; 1; q) &= G_{n,q}^{[m-1, \alpha]}(x, y; \lambda). \end{aligned} \quad (44)$$

Therefore,

$$\begin{aligned} &\lim_{q \rightarrow 1^-} \mathcal{T}_n^{[m-1, \alpha]}(x, y; \lambda; \mu; \nu; q) \\ &= \mathcal{T}_n^{[m-1, \alpha]}(x + y; \lambda, \mu; \nu), \end{aligned} \quad (45)$$

$$\lim_{q \rightarrow 1^-} \mathcal{T}_n^{[m-1, \alpha]}(0, 0; \lambda; \mu; \nu; q) = \mathcal{T}_n^{[m-1, \alpha]}(\lambda, \mu; \nu).$$

Clearly for $m = 1$, we have

$$\mathcal{T}_n^{[0, \alpha]}(x, y; \lambda; \mu; \nu; q) = \mathcal{T}_n^{(\alpha)}(x, y; \lambda; \mu; \nu; q). \quad (46)$$

For $\alpha = 1$, we have

$$\mathcal{T}_n^{[m-1, 1]}(x, y; \lambda; \mu; \nu; q) = \mathcal{T}_n^{[m-1]}(x, y; \lambda; \mu; \nu; q). \quad (47)$$

For $m = \alpha = 1$, we have

$$\mathcal{T}_n^{[0, 1]}(x, y; \lambda; \mu; \nu; q) = \mathcal{T}_n(x, y; \lambda; \mu; \nu; q). \quad (48)$$

Example 3. When $\alpha = 1, \mu = 0$, and $\nu = 1$ we can take $\lambda = -1$.

$$\begin{aligned} -\mathcal{T}_0^{[m-1,1]}(x, 0; -1; 0; 1; q) &= B_0^{[m-1]}(x; q) = [m]_q!, \\ -\mathcal{T}_1^{[m-1,1]}(x, 0; -1; 0; 1; q) &= B_1^{[m-1]}(x; q) \\ &= [m]_q! \left(x - \frac{1}{[m+1]_q} \right), \\ -\mathcal{T}_2^{[m-1,1]}(x, 0; -1; 0; 1; q) &= B_2^{[m-1]}(x; q) \\ &= [m]_q! \left(x^2 - \frac{[2]_q x}{[m+1]_q} + \frac{[2]_q q^{m+1}}{[m+2]_q [m+1]_q^2} \right). \end{aligned} \quad (49)$$

And the numbers are as follows:

$$\begin{aligned} -\mathcal{T}_0^{[m-1,1]}(0, 0; -1; 0; 1; q) &= B_0^{[m-1]}(q) = [m]_q!, \\ -\mathcal{T}_1^{[m-1,1]}(0, 0; -1; 0; 1; q) &= B_1^{[m-1]}(q) = -\frac{[m]_q!}{[m+1]_q}, \\ -\mathcal{T}_2^{[m-1,1]}(0, 0; -1; 0; 1; q) &= B_2^{[m-1]}(q) \\ &= \frac{[2]_q [m]_q! q^{m+1}}{[m+2]_q [m+1]_q^2}. \end{aligned} \quad (50)$$

Example 4. For any $\lambda \in \mathbb{C} \setminus \{-1, 1\}$, $m = 1, \alpha = 1, \mu = 1$, and $\nu = 2$

$$\begin{aligned} \mathcal{T}_j^{[0,1]}(x, 0; \lambda; 1; 2; q) &= 0, \quad j = 0, 1, \\ \mathcal{T}_2^{[0,1]}(x, 0; \lambda; 1; 2; q) &= \frac{2[2]_q}{\lambda + 1}, \\ \mathcal{T}_3^{[0,1]}(x, 0; \lambda; 1; 2; q) &= \frac{2[3]_q! x}{\lambda + 1} - \frac{2[3]_q! \lambda}{\lambda^2 + 2\lambda + 1}. \end{aligned} \quad (51)$$

Example 5. For any $\lambda \in \mathbb{C} \setminus \{-1, 1\}$, $m = 2, \alpha = 1, \mu = 3$, and $\nu = 1$

$$\begin{aligned} \mathcal{T}_j^{[1,1]}(x, 0; \lambda; 3; 1; q) &= 0, \quad j = 0, 1, \\ \mathcal{T}_2^{[1,1]}(x, 0; \lambda; 3; 1; q) &= \frac{64[2]_q}{\lambda + 1}, \\ \mathcal{T}_3^{[1,1]}(x, 0; \lambda; 3; 1; q) &= \frac{64[3]_q! x}{\lambda + 1} - \frac{64[3]_q!}{\lambda + 1}. \end{aligned} \quad (52)$$

The following proposition summarizes some properties of the polynomials $\mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q)$ which are a consequence of (40). Therefore, we will only show the details of its proofs (7) and (8).

Proposition 6. For a fixed $m \in \mathbb{N}$, $0 < |q| < 1$, let $\{\mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q)\}_{n=0}^\infty$ be the sequence of the q -analog of the generalized Apostol type polynomials in x, y , parameters $\alpha, \lambda, \mu, \nu \in \mathbb{C}$, order α , and level m . Then the following statements hold:

(1) *Special values:* for every $n \in \mathbb{N}_0$

$$\mathcal{T}_n^{[m-1,0]}(x, 0; \lambda; \mu; \nu; q) = x^n, \quad (53)$$

$$\mathcal{T}_n^{[m-1,0]}(0, y; \lambda; \mu; \nu; q) = q^{(1/2)n(n-1)} y^n. \quad (54)$$

(2) *Summation formulas:* for every $n \in \mathbb{N}_0$

$$\mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \quad (55)$$

$$\cdot \mathcal{T}_{n-k}^{[m-1,\alpha]}(\lambda; \mu; \nu; q) x^k,$$

$$\mathcal{T}_n^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \quad (56)$$

$$\cdot q^{(1/2)k(k-1)} \mathcal{T}_{n-k}^{[m-1,\alpha]}(\lambda; \mu; \nu; q) y^k,$$

$$\mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \quad (57)$$

$$\cdot \mathcal{T}_{n-k}^{[m-1,\alpha-1]}(\lambda; \mu; \nu; q) \mathcal{T}_k^{[m-1,1]}(x, 0; \lambda; \mu; \nu; q),$$

$$\mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \quad (58)$$

$$\cdot \mathcal{T}_{n-k}^{[m-1,\alpha]}(\lambda; \mu; \nu; q) \mathcal{T}_k^{[m-1,0]}(x, 0; \lambda; \mu; \nu; q).$$

(3) *Differential relations:* for $m \in \mathbb{N}$, fixed α, λ and $n, j \in \mathbb{N}_0$ with $0 \leq j \leq n$, we have

$$D_q \mathcal{T}_{n+1}^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) \quad (59)$$

$$= [n+1]_q \mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q),$$

$$D_q^{(j)} \mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) \quad (60)$$

$$= \frac{[n]_q!}{[n-j]_q!} \mathcal{T}_{n-j}^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q).$$

(4) *Integral formulas:* for $m \in \mathbb{N}$, fixed α, λ , we have

$$\begin{aligned} &\int_{x_0}^{x_1} \mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) d_q x \\ &= \frac{\mathcal{T}_{n+1}^{[m-1,\alpha]}(x_1, 0; \lambda; \mu; \nu; q) - \mathcal{T}_{n+1}^{[m-1,\alpha]}(x_0, 0; \lambda; \mu; \nu; q)}{[n+1]_q}, \end{aligned} \quad (61)$$

$$\int_{x_0}^{x_1} \mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) d_q x \quad (62)$$

$$= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_k^{[m-1,\alpha]}(\lambda; \mu; \nu; q) \left(\frac{x_1^{n+1-k} - x_0^{n+1-k}}{[n+1-k]_q} \right),$$

$$\mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) \quad (63)$$

$$= [n]_q \int_0^x \mathcal{T}_{n-1}^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) d_q x$$

$$+ \mathcal{T}_n^{[m-1,\alpha]}(\lambda; \mu; \nu; q).$$

(5) Addition theorems:

$$\mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \quad (64)$$

$$\cdot q^{(1/2)k(k-1)} \mathcal{T}_{n-k}^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) y^k,$$

$$\mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \quad (65)$$

$$\cdot q^{(1/2)k(k-1)} \mathcal{T}_{n-k}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) x^k,$$

$$\mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \quad (66)$$

$$\cdot \mathcal{T}_{n-k}^{[m-1,\alpha]}(\lambda; \mu; \nu; q) (x+y)_q^k,$$

$$\mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \quad (67)$$

$$\cdot \mathcal{T}_{n-k}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) x^k,$$

$$\mathcal{T}_n^{[m-1,\alpha+\beta]}(x, y; \lambda; \mu; \nu; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \quad (68)$$

$$\cdot \mathcal{T}_{n-k}^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) \mathcal{T}_k^{[m-1,\beta]}(0, y; \lambda; \mu; \nu; q).$$

Clearly, setting $x = 0$ in (64), we have

$$\begin{aligned} \mathcal{T}_n^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \\ = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(1/2)k(k-1)} \mathcal{T}_{n-k}^{[m-1,\alpha]}(\lambda; \mu; \nu; q) y^k. \end{aligned} \quad (69)$$

Setting $y = 0$ in (67), we obtain (55)

$$\begin{aligned} \mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) \\ = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(\lambda; \mu; \nu; q) x^k. \end{aligned} \quad (70)$$

Setting $y = 1$ and $x = 1$ in (64) and (67), respectively, we have

$$\begin{aligned} \mathcal{T}_n^{[m-1,\alpha]}(x, 1; \lambda; \mu; \nu; q) \\ = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(1/2)k(k-1)} \mathcal{T}_{n-k}^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q), \\ \mathcal{T}_n^{[m-1,\alpha]}(1, y; \lambda; \mu; \nu; q) \\ = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q). \end{aligned} \quad (71)$$

(6) If $a \in N$, we have

$$\begin{aligned} \mathcal{T}_n^{[m-1,\alpha]}(x, ax - x; \lambda; \mu; \nu; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \\ \cdot q^{(1/2)k(k-1)} \mathcal{T}_{n-k}^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) (a-1)^k x^k. \end{aligned} \quad (72)$$

(7) The q -analog of the generalized Apostol type polynomials satisfies the following relations:

$$\begin{aligned} \lambda \mathcal{T}_n^{[m-1,\alpha]}(1, y; \lambda; \mu; \nu; q) + \mathcal{T}_n^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \\ = 2 \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_k^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \end{aligned} \quad (73)$$

$$\times \mathcal{T}_{n-k}^{[0,-1]}(\lambda; 1; 0; q),$$

$$\begin{aligned} \lambda \mathcal{T}_n^{[m-1,\alpha]}(1, y; \lambda; \mu; \nu; q) \\ + \sum_{k=0}^{\min(n, m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \end{aligned} \quad (74)$$

$$= 2^{\mu m} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_k^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q)$$

$$\cdot \mathcal{T}_{n-k}^{[m-1,-1]}(\lambda; \mu; 0; q),$$

$$\begin{aligned} \lambda \mathcal{T}_n^{[m-1,\alpha]}(1, y; \lambda; \mu; \nu; q) \\ + \sum_{k=0}^{\min(n, m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) = 2^{\mu m} \end{aligned} \quad (75)$$

$$\cdot \frac{[n]_q!}{[n - \nu m]_q!} T_{n-\nu m}^{[m-1,\alpha-1]}(0, y; \lambda; \mu; \nu; q), \quad n \geq \nu m,$$

$$\begin{aligned} \lambda \mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) \\ + \sum_{k=0}^{\min(n, m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(x, -1; \lambda; \mu; \nu; q) = 2^{\mu m} \end{aligned} \quad (76)$$

$$\cdot \frac{[n]_q!}{[n - \nu m]_q!} T_{n-\nu m}^{[m-1,\alpha-1]}(x, -1; \lambda; \mu; \nu; q), \quad n \geq \nu m.$$

Proof (see (73)). Considering the expression $\lambda \mathcal{T}_n^{[m-1,\alpha]}(1, y; \lambda; \mu; \nu; q) + \mathcal{T}_n^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q)$ and using (40), we have

$$\begin{aligned} \mathbf{I} &:= \sum_{n=0}^{\infty} (\lambda \mathcal{T}_n^{[m-1,\alpha]}(1, y; \lambda; \mu; \nu; q) \\ &\quad + \mathcal{T}_n^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q)) \frac{z^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \lambda \mathcal{T}_n^{[m-1,\alpha]}(1, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\ &\quad + \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \end{aligned}$$

$$= \lambda \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^\alpha e_q^z E_q^{yz} \\ + \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^\alpha E_q^{yz}. \quad (77)$$

Now, factoring the previous expression, we get

$$\mathbf{I} = 2 \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^\alpha E_q^{yz} \left(\frac{\lambda e_q^z + 1}{2} \right) \\ = 2 \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1, \alpha]}(0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\ \cdot \sum_{n=0}^{\infty} \mathcal{T}_n^{[0, -1]}(\lambda; 1; 0; q) \frac{z^n}{[n]_q!} = \sum_{n=0}^{\infty} 2 \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \\ \cdot \mathcal{T}_k^{[m-1, \alpha]}(0, y; \lambda; \mu; \nu; q) \mathcal{T}_{n-k}^{[0, -1]}(\lambda; 1; 0; q) \frac{z^n}{[n]_q!}. \quad (78)$$

Comparing the coefficients of $z^n / [n]_q!$ in both sides gives the result.

This completes the proof.

Proof (see (74)). By using the relation (see [23, p.5, Lemma 6])

$$\sum_{l=0}^{m-1} \frac{z^l}{[l]_q!} \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1, \alpha]}(0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\ = \sum_n \sum_{k=0}^{\min(n, m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1, \alpha]}(0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \quad (79)$$

and (40), we have

$$\sum_{n=0}^{\infty} \left(\lambda \mathcal{T}_n^{[m-1, \alpha]}(1, y; \lambda; \mu; \nu; q) \right. \\ \left. + \sum_{k=0}^{\min(n, m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1, \alpha]}(0, y; \lambda; \mu; \nu; q) \right) \frac{z^n}{[n]_q!} \\ = \sum_{n=0}^{\infty} \lambda \mathcal{T}_n^{[m-1, \alpha]}(1, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} + \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!} \\ \cdot \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1, \alpha]}(0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\ = \lambda \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^\alpha e_q^z E_q^{yz} + \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!} \\ \cdot \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^\alpha E_q^{yz} \quad (80)$$

Now, factoring the above expression, we get

$$\sum_{n=0}^{\infty} \left(\lambda \mathcal{T}_n^{[m-1, \alpha]}(1, y; \lambda; \mu; \nu; q) \right. \\ \left. + \sum_{k=0}^{\min(n, m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1, \alpha]}(0, y; \lambda; \mu; \nu; q) \right) \frac{z^n}{[n]_q!} \\ = 2^{\mu m} \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^\alpha \\ \cdot E_q^{yz} \left(\frac{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)}{2^{\mu m}} \right) \\ = 2^{\mu m} \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1, \alpha]}(0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\ \cdot \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1, -1]}(\lambda; \mu; 0; q) \frac{z^n}{[n]_q!} = 2^{\mu m} \sum_{n=0}^{\infty} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \\ \cdot \mathcal{T}_k^{[m-1, \alpha]}(0, y; \lambda; \mu; \nu; q) \mathcal{T}_{n-k}^{[m-1, -1]}(\lambda; \mu; 0; q) \\ \cdot \frac{z^n}{[n]_q!}. \quad (81)$$

Comparing the coefficients of $z^n / [n]_q!$ in both sides gives the result.

This completes the proof.

Proof (see (75)). Let

$$\mathbf{I} := \sum_{n=0}^{\infty} \left(\lambda \mathcal{T}_n^{[m-1, \alpha]}(1, y; \lambda; \mu; \nu; q) \right. \\ \left. + \sum_{k=0}^{\min(n, m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1, \alpha]}(0, y; \lambda; \mu; \nu; q) \right) \frac{z^n}{[n]_q!}. \quad (82)$$

Using (40) and (79), we have

$$\mathbf{I} = \sum_{n=0}^{\infty} \lambda \mathcal{T}_n^{[m-1, \alpha]}(1, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\ + \sum_{n=0}^{\infty} \sum_{k=0}^{\min(n, m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1, \alpha]}(0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\ = \sum_{n=0}^{\infty} \lambda \mathcal{T}_n^{[m-1, \alpha]}(1, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} + \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!} \\ \cdot \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1, \alpha]}(0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!}$$

$$\begin{aligned}
&= \lambda \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^\alpha e_q^z E_q^{yz} + \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!} \\
&\cdot \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^\alpha E_q^{yz} = (2^\mu z^\nu)^m \\
&\cdot \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^\alpha \\
&\cdot E_q^{yz} \left(\frac{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)}{(2^\mu z^\nu)^m} \right) \\
&= 2^{\mu m} z^{\nu m} \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^{\alpha-1} E_q^{yz}.
\end{aligned} \tag{83}$$

Therefore, we get

$$\begin{aligned}
\mathbf{I} &= 2^{\mu m} \sum_{n=0}^{\infty} \mathcal{F}_n^{[m-1, \alpha-1]}(0, y; \lambda; \mu; \nu; q) \frac{z^{n+\nu m}}{[n]_q!} \\
&= 2^{\mu m} \sum_{n=0}^{\infty} \frac{[n + \nu m]_q!}{[n]_q!} \mathcal{F}_n^{[m-1, \alpha-1]}(0, y; \lambda; \mu; \nu; q) \\
&\cdot \frac{z^{n+\nu m}}{[n + \nu m]_q!} = 2^{\mu m} \sum_{n-\nu m=0}^{\infty} \frac{[n]_q!}{[n - \nu m]_q!} \\
&\cdot \mathcal{F}_{n-\nu m}^{[m-1, \alpha-1]}(0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!}.
\end{aligned} \tag{84}$$

Notice that

$$\begin{aligned}
&\mathcal{F}_0^{[m-1, \alpha-1]}(0, y; \lambda; \mu; \nu; q) \\
&= 0, \dots, \mathcal{F}_{n-1-\nu m}^{[m-1, \alpha-1]}(0, y; \lambda; \mu; \nu; q) = 0.
\end{aligned} \tag{85}$$

Comparing the coefficients of $z^n / [n]_q!$ in both sides gives the result.

This completes the proof.

Proof (see (76)). Let

$$\begin{aligned}
\mathbf{I} &:= \sum_{n=0}^{\infty} \left(\lambda \mathcal{F}_n^{[m-1, \alpha]}(x, 0; \lambda; \mu; \nu; q) \right. \\
&+ \sum_{k=0}^{\min(n, m-1)} \left[\begin{matrix} n \\ k \end{matrix} \right]_q \mathcal{F}_{n-k}^{[m-1, \alpha]}(x, -1; \lambda; \mu; \nu; q) \Big) \\
&\cdot \frac{z^n}{[n]_q!}.
\end{aligned} \tag{86}$$

Using (40) and (79), we have

$$\begin{aligned}
\mathbf{I} &= \sum_{n=0}^{\infty} \lambda \mathcal{F}_n^{[m-1, \alpha]}(x, 0; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\
&+ \sum_{n=0}^{\infty} \sum_{k=0}^{\min(n, m-1)} \left[\begin{matrix} n \\ k \end{matrix} \right]_q \mathcal{F}_{n-k}^{[m-1, \alpha]}(x, -1; \lambda; \mu; \nu; q) \\
&\cdot \frac{z^n}{[n]_q!} = \sum_{n=0}^{\infty} \lambda \mathcal{F}_n^{[m-1, \alpha]}(x, 0; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\
&+ \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!} \sum_{n=0}^{\infty} \mathcal{F}_n^{[m-1, \alpha]}(x, -1; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\
&= \lambda \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^\alpha e_q^{xz} + \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!} \\
&\cdot \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^\alpha e_q^{xz} E_q^{-z}.
\end{aligned} \tag{87}$$

then, factoring the above equation and using (17), we have

$$\begin{aligned}
\mathbf{I} &= 2^{\mu m} z^{\nu m} \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^\alpha \\
&\cdot e_q^{xz} \left(\frac{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)}{(2^\mu z^\nu)^m} \right) \cdot \frac{1}{e_q^z} \\
&= 2^{\mu m} z^{\nu m} \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^{\alpha-1} e_q^{xz} E_q^{-z}.
\end{aligned} \tag{88}$$

Therefore, we get

$$\begin{aligned}
\mathbf{I} &= 2^{\mu m} \sum_{n=0}^{\infty} \mathcal{F}_n^{[m-1, \alpha-1]}(x, -1; \lambda; \mu; \nu; q) \frac{z^{n+\nu m}}{[n]_q!} \\
&= 2^{\mu m} \sum_{n=0}^{\infty} \frac{[n + \nu m]_q!}{[n]_q!} \mathcal{F}_n^{[m-1, \alpha-1]}(x, -1; \lambda; \mu; \nu; q) \\
&\cdot \frac{z^{n+\nu m}}{[n + \nu m]_q!} = 2^{\mu m} \sum_{n-\nu m=0}^{\infty} \frac{[n]_q!}{[n - \nu m]_q!} \\
&\cdot \mathcal{F}_{n-\nu m}^{[m-1, \alpha-1]}(x, -1; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!}.
\end{aligned} \tag{89}$$

Notice that

$$\begin{aligned}
&\mathcal{F}_0^{[m-1, \alpha-1]}(x, -1; \lambda; \mu; \nu; q) \\
&= 0, \dots, \mathcal{F}_{n-1-\nu m}^{[m-1, \alpha-1]}(x, -1; \lambda; \mu; \nu; q) = 0.
\end{aligned} \tag{90}$$

Comparing the coefficients of $z^n / [n]_q!$ in both sides gives the desired result.

This completes the proof.

Remarks 7. Setting $\mu = 1$, $\nu = 0$, $\lambda = 1$ in (75), we obtain

$$\begin{aligned} & \mathcal{T}_n^{[m-1,\alpha]}(1, y; 1; 1; 0; q) \\ & + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(0, y; 1; 1; 0; q) \\ & = 2^m \mathcal{T}_n^{[m-1,\alpha-1]}(0, y; 1; 1; 0; q), \end{aligned} \quad (91)$$

Note that (91) is equivalent to [23, Lemma 6, Eq.2].

Substituting $\nu = 0$ in (75), we obtain

$$\begin{aligned} & \lambda \mathcal{T}_n^{[m-1,\alpha]}(1, y; \lambda; \mu; 0; q) \\ & + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(0, y; \lambda; \mu; 0; q) \\ & = 2^{\mu m} \mathcal{T}_n^{[m-1,\alpha-1]}(0, y; \lambda; \mu; 0; q). \end{aligned} \quad (92)$$

(8) For $n \in \mathbb{N}_0$, $n \geq \nu m$, we have

$$\begin{aligned} y^{n-\nu m} & = \frac{[n - \nu m]_q!}{2^{\mu m} [n]_q! q^{(1/2)(n-\nu m)(n-\nu m-1)}} \\ & \times \left(\lambda \mathcal{T}_n^{[m-1,1]}(1, y; \lambda; \mu; \nu; q) \right. \\ & \left. + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,1]}(0, y; \lambda; \mu; \nu; q) \right). \end{aligned} \quad (93)$$

Proof (see (93)). Putting $\alpha = 1$ in (75) and using (54), we obtain

$$\begin{aligned} & \lambda \mathcal{T}_n^{[m-1,1]}(1, y; \lambda; \mu; \nu; q) \\ & + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,1]}(0, y; \lambda; \mu; \nu; q) \\ & = 2^{\mu m} \frac{[n]_q!}{[n - \nu m]_q!} \mathcal{T}_{n-\nu m}^{[m-1,0]}(0, y; \lambda; \mu; \nu; q) \\ & = 2^{\mu m} \frac{[n]_q!}{[n - \nu m]_q!} q^{(1/2)(n-\nu m)(n-\nu m-1)} y^{n-\nu m}, \end{aligned} \quad (94)$$

then

$$\begin{aligned} y^{n-\nu m} & = \frac{[n - \nu m]_q!}{2^{\mu m} [n]_q! q^{(1/2)(n-\nu m)(n-\nu m-1)}} \\ & \times \left(\lambda \mathcal{T}_n^{[m-1,1]}(1, y; \lambda; \mu; \nu; q) \right. \\ & \left. + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,1]}(0, y; \lambda; \mu; \nu; q) \right). \end{aligned} \quad (95)$$

This completes the proof.

4. Some Connection Formulas for the Polynomials $\mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q)$

In this section, we introduce some formulas of connection between the q -analog of the generalized Apostol type polynomials $\mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q)$ and the generalized q -Bernoulli polynomials of level m , the q -Stirling numbers of the second kind, the generalized q -Apostol type Bernoulli, q -Apostol type Euler, q -Apostol type Genocchi polynomials of order α and level m , and the q -Bernstein polynomials.

Proposition 8. For $n, j, k \in \mathbb{N}_0$, $m \in \mathbb{N}$, and $0 < |q| < 1$, the q -analog of the generalized Apostol type polynomials of level m is related to the generalized q -Bernoulli polynomials of level m and the q -gamma function

$$\begin{aligned} & \mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q) \\ & = \sum_{j=0}^n \sum_{k=0}^j \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q \frac{[k]_q!}{[k+m]_q!} \\ & \cdot q^{(1/2)j(j-1)} \mathcal{T}_{n-j}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \\ & \cdot B_{j-k}^{[m-1,1]}(x, 0; q), \end{aligned} \quad (96)$$

$$\begin{aligned} & \mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q) = \sum_{j=0}^n \sum_{k=0}^j \begin{bmatrix} n \\ j \end{bmatrix}_q \\ & \cdot \frac{[j]_q! q^{(1/2)j(j-1)}}{[k]_q!} \\ & \cdot \frac{\mathcal{T}_{n-j}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) B_k^{[m-1,1]}(x, 0; q)}{\Gamma_q(j-k+m+1)}, \end{aligned} \quad (97)$$

$$\begin{aligned} & \mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q) = \sum_{j=0}^n \sum_{k=0}^j \begin{bmatrix} n \\ j \end{bmatrix}_q \\ & \cdot \frac{[j]_q! \mathcal{T}_{n-j}^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) B_k^{[m-1,1]}(0, y; q)}{[j+m-k]_q! [k]_q!}. \end{aligned} \quad (98)$$

Proof. We only prove (97). Substituting (36) into the right-hand side of (65), we have

$$\begin{aligned} & \mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q) = \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q \\ & \cdot q^{(1/2)j(j-1)} \mathcal{T}_{n-j}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \\ & \cdot \sum_{k=0}^j \frac{[j]_q! B_k^{[m-1,1]}(x, 0; q)}{[k]_q! \Gamma_q(j-k+m+1)} = \sum_{j=0}^n \sum_{k=0}^j \begin{bmatrix} n \\ j \end{bmatrix}_q \\ & \cdot q^{(1/2)j(j-1)} \mathcal{T}_{n-j}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \end{aligned}$$

$$\begin{aligned}
& \cdot \frac{[j]_q! B_k^{[m-1,1]}(x, 0; q)}{[k]_q! \Gamma_q(j-k+m+1)} = \sum_{j=0}^n \sum_{k=0}^j \begin{bmatrix} n \\ j \end{bmatrix}_q \\
& \cdot \frac{[j]_q! q^{(1/2)j(j-1)}}{[k]_q!} \\
& \cdot \frac{\mathcal{T}_{n-j}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) B_k^{[m-1,1]}(x, 0; q)}{\Gamma_q(j-k+m+1)}.
\end{aligned} \quad (99)$$

□

Proposition 9. For $n, j, k \in \mathbb{N}_0$, $m \in \mathbb{N}$, and $0 < |q| < 1$, the q -analog of the generalized Apostol type polynomials is related to the q -Stirling numbers of the second kind $S(n, k; q)$ by means the following identities:

$$\begin{aligned}
& \mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q) = \sum_{j=0}^n \sum_{k=0}^j \begin{bmatrix} n \\ j \end{bmatrix}_q \\
& \cdot q^{(1/2)j(j-1)} \mathcal{T}_{n-j}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) S(j, k; q) \\
& \cdot (x)_{q;k},
\end{aligned} \quad (100)$$

$$\begin{aligned}
& \mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) = \sum_{j=0}^n \sum_{k=0}^j \begin{bmatrix} n \\ j \end{bmatrix}_q \\
& \cdot \mathcal{T}_{n-j}^{[m-1,\alpha]}(\lambda; \mu; \nu; q) S(j, k; q) (x)_{q;k}.
\end{aligned} \quad (101)$$

Proof. Substituting (26) into the right-hand side of (55) and (65) gives the results. □

Proposition 10. For $n, k \in \mathbb{N}_0$, $m \in \mathbb{N}$, and $0 < |q| < 1$, the q -analog of the generalized Apostol type polynomials is related to the generalized q -Apostol-Bernoulli polynomials, the generalized q -Apostol-Euler polynomials, and the generalized q -Apostol-Genocchi polynomials by means the following identities:

$$\begin{aligned}
& \lambda \mathcal{T}_n^{[m-1,\alpha]}(1, y; \lambda; \mu; \nu; q) \\
& - \sum_{k=0}^{\min(n, m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \\
& = \sum_{k=0}^{n-m} \begin{bmatrix} n-m \\ k \end{bmatrix}_q \frac{[n]_q!}{[n-m]_q!} \\
& \cdot \mathcal{T}_k^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \mathfrak{B}_{n-m-k}^{[m-1,-1]}(\lambda; q), \\
& \quad n \geq m,
\end{aligned} \quad (102)$$

$$\begin{aligned}
& \lambda \mathcal{T}_n^{[m-1,\alpha]}(1, y; \lambda; \mu; \nu; q) \\
& + \sum_{k=0}^{\min(n, m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \\
& = 2^m \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_k^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \frac{z^k}{[k]_q!} \\
& \cdot \mathfrak{E}_{n-k}^{[m-1,-1]}(\lambda; q),
\end{aligned} \quad (103)$$

$$\begin{aligned}
& \lambda \mathcal{T}_n^{[m-1,\alpha]}(1, y; \lambda; \mu; \nu; q) + \sum_{k=0}^{\min(n, m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \\
& \cdot \mathcal{T}_{n-k}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) = 2^m \sum_{k=0}^{n-m} \begin{bmatrix} n-m \\ k \end{bmatrix}_q \\
& \cdot \frac{[n]_q!}{[n-m]_q!} \mathcal{T}_k^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \\
& \cdot \mathfrak{G}_{n-m-k}^{[m-1,-1]}(\lambda; q), \quad n \geq m.
\end{aligned} \quad (104)$$

We will only show the details of the demonstrations of (102) and (103).

Proof (see (102)). Using (40) and (79), we have

$$\begin{aligned}
\mathbf{I} &:= \sum_{n=0}^{\infty} \left(\lambda \mathcal{T}_n^{[m-1,\alpha]}(1, y; \lambda; \mu; \nu; q) \right. \\
& \left. - \sum_{k=0}^{\min(n, m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \right) \frac{z^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \lambda \mathcal{T}_n^{[m-1,\alpha]}(1, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} - \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!} \\
& \cdot \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\
&= \lambda \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^\alpha e_q^z E_q^{yz} - \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!} \\
& \cdot \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^\alpha E_q^{yz}.
\end{aligned} \quad (105)$$

Now, factoring the above equation and using (7), (40), we get

$$\begin{aligned}
\mathbf{I} &= z^m \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^\alpha \\
& \cdot E_q^{yz} \left(\frac{\lambda e_q^z - \sum_{l=0}^{m-1} (z^l / [l]_q!)}{z^m} \right) \\
&= \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!}
\end{aligned}$$

$$\begin{aligned}
 & \cdot \sum_{n=0}^{\infty} \mathfrak{B}_n^{[m-1,-1]}(\lambda; q) \frac{z^{n+m}}{[n]_q!} \\
 &= \sum_{n=0}^{\infty} \sum_{k=0}^n \mathcal{T}_k^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \frac{z^k}{[k]_q!} \\
 & \cdot \mathfrak{B}_{n-k}^{[m-1,-1]}(\lambda; q) \frac{z^{n-k+m}}{[n-k]_q!}.
 \end{aligned} \tag{106}$$

Then, we get

$$\begin{aligned}
 \mathbf{I} &= \sum_{n=m}^{\infty} \sum_{k=0}^{n-m} \mathcal{T}_k^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \frac{z^k}{[k]_q!} \\
 & \cdot \mathfrak{B}_{n-m-k}^{[m-1,-1]}(\lambda; q) \frac{z^{n-k}}{[n-m-k]_q!} \\
 &= \sum_{n=m}^{\infty} \sum_{k=0}^{n-m} \begin{bmatrix} n-m \\ k \end{bmatrix}_q \frac{[n]_q!}{[n-m]_q!} \\
 & \cdot \mathcal{T}_k^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \mathfrak{B}_{n-m-k}^{[m-1,-1]}(\lambda; q) \frac{z^n}{[n]_q!}.
 \end{aligned} \tag{107}$$

Comparing the coefficients of $z^n/[n]_q!$ in both sides gives the result. \square

Proof (see (103)). Using (40) and (79), we have

$$\begin{aligned}
 \mathbf{I} &:= \sum_{n=0}^{\infty} \left(\lambda \mathcal{T}_n^{[m-1,\alpha]}(1, y; \lambda; \mu; \nu; q) \right. \\
 & \quad \left. + \sum_{k=0}^{\min(n, m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \right) \frac{z^n}{[n]_q!} \\
 &= \sum_{n=0}^{\infty} \lambda \mathcal{T}_n^{[m-1,\alpha]}(1, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} + \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!} \\
 & \cdot \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\
 &= \lambda \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l/[l]_q!)} \right)^\alpha e_q^z E_q^{yz} + \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!} \\
 & \cdot \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l/[l]_q!)} \right)^\alpha E_q^{yz}.
 \end{aligned} \tag{108}$$

Now, factoring the previous equation and using (8), (40), we get

$$\begin{aligned}
 \mathbf{I} &= 2^m \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l/[l]_q!)} \right)^\alpha \\
 & \cdot E_q^{yz} \left(\frac{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l/[l]_q!)}{2^m} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= 2^m \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\
 & \cdot \sum_{n=0}^{\infty} \mathfrak{B}_n^{[m-1,-1]}(\lambda; q) \frac{z^n}{[n]_q!} \\
 &= 2^m \sum_{n=0}^{\infty} \sum_{k=0}^n \mathcal{T}_k^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \frac{z^k}{[k]_q!} \\
 & \cdot \mathfrak{B}_{n-k}^{[m-1,-1]}(\lambda; q) \frac{z^{n-k}}{[n-k]_q!}.
 \end{aligned} \tag{109}$$

Therefore, we get

$$\begin{aligned}
 \mathbf{I} &= \sum_{n=0}^{\infty} 2^m \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_k^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \frac{z^k}{[k]_q!} \\
 & \cdot \mathfrak{B}_{n-k}^{[m-1,-1]}(\lambda; q) \frac{z^n}{[n]_q!}.
 \end{aligned} \tag{110}$$

Comparing the coefficients of $z^n/[n]_q!$ in both sides gives the result. \square

Proposition 11. For $n, j, k \in \mathbb{N}_0$, $m \in \mathbb{N}$, and $0 < |q| < 1$, the q -analog of the generalized Apostol type polynomials is related to q -Bernstein basis $B_k^n(x; q)$ by means of the following identities:

$$\begin{aligned}
 \mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) &= \sum_{j=0}^n \sum_{k=0}^{n-j} \begin{bmatrix} k+j \\ j \end{bmatrix}_q \\
 & \cdot \mathcal{T}_{n-j}^{[m-1,\alpha]}(\lambda; \mu; \nu; q) B_{k+j}^n(x; q),
 \end{aligned} \tag{111}$$

$$\begin{aligned}
 \mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q) &= \sum_{j=0}^n \sum_{k=0}^{n-j} \begin{bmatrix} k+j \\ j \end{bmatrix}_q \\
 & \cdot q^{(1/2)j(j-1)} \mathcal{T}_{n-j}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) B_{k+j}^n(x; q).
 \end{aligned} \tag{112}$$

Proof (see (111)). Substituting (31) into (55) we have

$$\begin{aligned}
 \mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q \mathcal{T}_{n-j}^{[m-1,\alpha]}(\lambda; \mu; \nu; q) \sum_{k=j}^n \begin{bmatrix} k \\ j \end{bmatrix}_q B_k^n(x; q) \\
 &= \sum_{j=0}^n \sum_{k=j}^n \begin{bmatrix} k \\ j \end{bmatrix}_q \mathcal{T}_{n-j}^{[m-1,\alpha]}(\lambda; \mu; \nu; q) B_k^n(x; q) \\
 &= \sum_{j=0}^n \sum_{k=0}^{n-j} \begin{bmatrix} k+j \\ j \end{bmatrix}_q \mathcal{T}_{n-j}^{[m-1,\alpha]}(\lambda; \mu; \nu; q) B_{k+j}^n(x; q).
 \end{aligned} \tag{113}$$

\square

Proof (see (112)). Substituting (31) into (65) we have

$$\begin{aligned}
 \mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q) &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q \\
 &\cdot q^{(1/2)(j)(j-1)} \mathcal{T}_{n-j}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \sum_{k=j}^n \frac{\begin{bmatrix} k \\ j \end{bmatrix}_q}{\begin{bmatrix} n \\ j \end{bmatrix}_q} \\
 &\cdot B_k^n(x; q) = \sum_{j=0}^n \sum_{k=j}^n \begin{bmatrix} k \\ j \end{bmatrix}_q \\
 &\cdot q^{(1/2)(j)(j-1)} \mathcal{T}_{n-j}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) B_k^n(x; q) \\
 &= \sum_{j=0}^n \sum_{k=0}^{n-j} \begin{bmatrix} k+j \\ j \end{bmatrix}_q \\
 &\cdot q^{(1/2)(j)(j-1)} \mathcal{T}_{n-j}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) B_{k+j}^n(x; q).
 \end{aligned} \tag{114}$$

□

Proposition 12. For $n, k, j \in \mathbb{N}_0$, $m \in \mathbb{N}$, and $0 < |q| < 1$

$$\begin{aligned}
 B_k^n(x; q) &= x^k \sum_{j=0}^{n-k} \begin{bmatrix} n-k \\ j \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k-j}^{[m-1,-1]}(\lambda; \mu; \nu; q) \\
 &\cdot \mathcal{T}_j^{[m-1,1]}(1, -x; \lambda; \mu; \nu; q).
 \end{aligned} \tag{115}$$

Proof (see (115)). The q -Bernstein basis is defined by means of following generating function:

$$\frac{x^k z^k}{[k]_q!} e_q^z E_q^{-xz} = \sum_{n=k}^{\infty} B_k^n(x; q) \frac{z^n}{[n]_q!}. \tag{116}$$

Using the left-hand side of the previous equation and (40), we have

$$\begin{aligned}
 \frac{x^k z^k}{[k]_q!} e_q^z E_q^{-xz} &= \frac{x^k z^k}{[k]_q!} \left(\frac{e_q^z - \sum_{h=0}^{m-1} (z^h / [h]_q!)}{(2^\mu z^\nu)^m} \right) \\
 &\cdot \left(\frac{(2^\mu z^\nu)^m}{e_q^z - \sum_{h=0}^{m-1} (z^h / [h]_q!)} \right) e_q^z E_q^{-xz} = \frac{x^k z^k}{[k]_q!} \\
 &\cdot \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1,-1]}(\lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\
 &\cdot \sum_{j=0}^{\infty} \mathcal{T}_j^{[m-1,1]}(1, -x; \lambda; \mu; \nu; q) \frac{z^j}{[j]_q!} = \frac{x^k}{[k]_q!} \\
 &\cdot \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1,-1]}(\lambda; \mu; \nu; q) \frac{z^{n+k}}{[n]_q!} \\
 &\cdot \sum_{j=0}^{\infty} \mathcal{T}_j^{[m-1,1]}(1, -x; \lambda; \mu; \nu; q) \frac{z^j}{[j]_q!}.
 \end{aligned} \tag{117}$$

Therefore, we obtain

$$\begin{aligned}
 \frac{x^k z^k}{[k]_q!} e_q^z E_q^{-xz} &= \frac{x^k}{[k]_q!} \sum_{n=k}^{\infty} \sum_{j=0}^{n-k} \mathcal{T}_{n-k-j}^{[m-1,-1]}(\lambda; \mu; \nu; q) \\
 &\cdot \frac{z^{n-j}}{[n-k-j]_q!} \mathcal{T}_j^{[m-1,1]}(1, -x; \lambda; \mu; \nu; q) \frac{z^j}{[j]_q!} \\
 &= \sum_{n=k}^{\infty} x^k \sum_{j=0}^{n-k} \begin{bmatrix} n-k \\ j \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k-j}^{[m-1,-1]}(\lambda; \mu; \nu; q) \\
 &\cdot \mathcal{T}_j^{[m-1,1]}(1, -x; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!}.
 \end{aligned} \tag{118}$$

Comparing coefficients the $z^n/[n]_q!$, we obtain

$$\begin{aligned}
 B_k^n(x; q) &= x^k \sum_{j=0}^{n-k} \begin{bmatrix} n-k \\ j \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k-j}^{[m-1,-1]}(\lambda; \mu; \nu; q) \\
 &\cdot \mathcal{T}_j^{[m-1,1]}(1, -x; \lambda; \mu; \nu; q).
 \end{aligned} \tag{119}$$

□

Corollary 13. For $n, j, k \in \mathbb{N}_0$, $0 \leq k \leq j \leq n$, $m \in \mathbb{N}$, and $0 < |q| < 1$, one has

$$\begin{aligned}
 \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q S(j, k; q) \mathfrak{B}_{n-j}^{(k)}(1, -x; q) \\
 = \sum_{j=0}^{n-k} \begin{bmatrix} n-k \\ j \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k-j}^{[m-1,-1]}(\lambda; \mu; \nu; q) \\
 \cdot \mathcal{T}_j^{[m-1,1]}(1, -x; \lambda; \mu; \nu; q).
 \end{aligned} \tag{120}$$

Proof (see (120)). Substituting (32) into the left-hand side of (115), we obtain the result. □

Data Availability

No data were used to support this study.

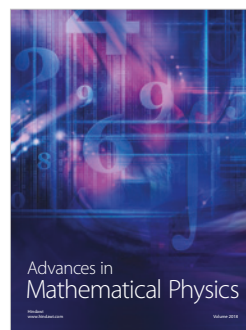
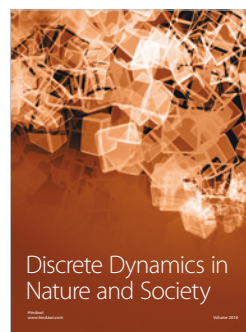
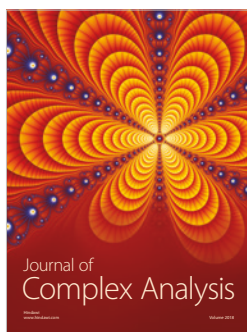
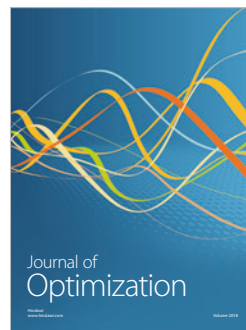
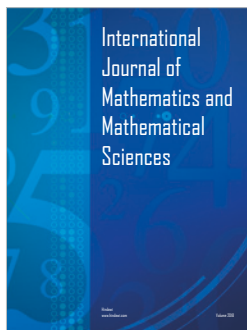
Conflicts of Interest

The authors declare that they have no conflicts of interest.

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